1. $t=0$: long put 90   long $2-2\times100$ cash
long put 110   long 2 stocks
short $2\times$ call 100

(idea: $P_E(t=0, K=90, T=2) + P_E(t=0, K=110, T=2)$

$\geq 2P_E(t=0, K=100, T=2)$ (convexity)

$\geq 2P_E(t=0, K=100, T=1)$ (comparison principle)

$= 2(C_E(t=0, K=100, T=1) - S_0 + 100)$ (put-call parity)

but $6+14 < 2(11-100+100)$.)

Then, at $t=0$, $6+14-2\times11+2-2\times100+2\times100=0$.

$t=1$: $P_E(t=1, K=90, T=2) + P_E(t=1, K=110, T=2)$

$-2(S_1-100)^+ + 2-2\times100+2S_1$

$\geq P_E(t=1, K=90, T=1) + P_E(t=1, K=110, T=1) - 2(S_1-100)^+$
$+2-2\times100+2S_1$

$\geq 2P_E(t=1, K=100, T=1) - 2(S_1-100)^+ + 2-2\times100+2S_1$

$= 2(100-S_1)^+ - 2(S_1-100)^+ + 2 + 2(S_1-100)$

$$= 2\cancel{(S_1-100)^-} - 2\cancel{(S_1-100)^+} + 2 + 2\cancel{(S_1-100)^+} - 2\cancel{(S_1-100)^-}$$

$$> 0.$$

2. $r_1 = 0.02$, $r_2 = 0.03$. Present values are Identical.

$$\sum_{j=1}^{60} \frac{12000}{(1+\frac{r_1}{4})^j} = \sum_{j=1}^{360} \frac{x}{(1+\frac{r_2}{12})^j} \quad \Rightarrow x \approx 1441.42$$

3. (1) $f(t, B_t) = tB_t$

$$d(tB_t) = B_t\,dt + t\,dB_t$$

$$\Rightarrow tB_t = \int_0^t B_s\,ds + \int_0^t s\,dB_s.$$

(2) $$\int_0^t B_s\,ds = tB_t - \int_0^t s\,dB_s$$

$$= t\int_0^t dB_s - \int_0^t s\,dB_s$$

$$= \int_0^t (t-s)\,dB_s.$$

(3) $$E\left[\int_0^t B_s\,ds\right] = E\left[\int_0^t (t-s)\,dB_s\right] = 0$$

$$E\left[\left(\int_0^t B_s\,ds\right)^2\right] = E\left[\left(\int_0^t (t-s)\,dB_s\right)^2\right]$$

$$= E\left[\int_0^t (t-s)^2 ds\right]$$

$$= \int_0^t (t^2+s^2-2ts)\,ds$$

$$= t^2 \cdot t + \frac{t^3}{3} - t \cdot t^2 = \frac{t^3}{3}$$

$$Var\left(\int_0^t B_s ds\right) = E\left(\left(\int_0^t B_s ds\right)^2\right) - \left(E\left(\int_0^t B_s ds\right)\right)^2$$

$$= \frac{t^3}{3}$$

4. (a) self-financing $\Rightarrow$ $d\Pi_t^{x,\phi} = (\Pi_t - \phi_t S_t) r dt + \phi_t dS_t$

$$= (\Pi_t - \phi_t S_t) r dt + \phi_t (\mu S_t dt + 6 S_t dB_t)$$

$$= ((\Pi_t - \phi_t S_t) r + \phi_t \mu S_t) dt$$

$$+ 6 \phi_t S_t dB_t$$

(b). $S_t = S_0 \exp\left((r - \frac{6^2}{2})t + 6 B_t^Q\right)$

(c) $v_0 = E^Q[e^{-rT} S_T^2]$

$$= E^Q\left[e^{-rT} \cdot S_0^2 \exp\left(2(r - \frac{6^2}{2})T + 26 B_T^Q\right)\right]$$

$$= e^{-rT} S_0^2 \cdot \exp\left(2(r - \frac{6^2}{2})T\right) \cdot \exp\left(\frac{46^2}{2} T\right)$$

$$= S_0^2 \exp(rT + 6^2 T)$$

$$d\left(e^{-rt}v(t,S_t)\right) = -re^{-rt}v(t,S_t)dt + e^{-rt}dv(t,S_t)$$

$$dv(t,S_t) = \partial_t v\,dt + \partial_x v\,dS_t + \frac{\partial_{xx}v}{2}\sigma^2 S_t^2\,dt$$

Then, $d\left(e^{-rt}v(t,S_t)\right) = e^{-rt}\cdot 2\sigma\cdot S_t^2\cdot \exp\left((r+\sigma^2)(T-t)\right)$
dB_t.

Notice that $d\tilde{S}_t = d(e^{-rt}S_t) = e^{-rt}\sigma S_t dB_t$.

Thus, $d\left(e^{-rt}v(t,S_t)\right) = 2S_t\exp\left((r+\sigma^2)(T-t)\right)d\tilde{S}_t$

$$\left(\partial_x v(t,S_t) = 2S_t\exp\left((r+\sigma^2)(T-t)\right)\right)$$

$$= \partial_x v(t,S_t)\,d\tilde{S}_t.$$

$$\Rightarrow e^{-rT}v(T,S_T) - v(0,S_0) = \int_0^T \partial_x v(t,S_t)\,d\tilde{S}_t$$

$$e^{-rT}S_T^2 - S_0^2\exp\left((r+\sigma^2)T\right) = \int_0^T \partial_x v(t,S_t)\,d\tilde{S}_t$$

$$g(S_T) = V_0 + \int_0^T \partial_x v(t,S_t)\,d\tilde{S}_t.$$